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Covariant Material Equations for Liquids in Electromagnetic Fields

The equations of balance for materials without internal variables in special relativity read [2], [3]:

(Balance of particle number) $$(n_0 u^\alpha)_{,\alpha} = 0 \tag{1}$$

(Balance of energy-momentum) $$T^{\alpha\beta}_{,\beta} + \underset{\text{em}}{T^{\alpha\beta}_{,\beta}} = 0 \tag{2}$$

(Blance of moment of momentum) $$T^{[\alpha\beta]} + \underset{\text{em}}{T^{[\alpha\beta]}} = 0 \tag{3}$$

(Maxwell's equations) $$\varepsilon^{\alpha\beta\gamma\mu} F_{\beta\gamma,\mu} = 0 \tag{4}$$

$$G^{\alpha\beta}_{,\beta} = j^\alpha , \tag{5}$$

where the mechanical energy-momentum tensor is given by

$$T^{\alpha\beta} = \frac{1}{c^2} (n_0 e u^\alpha u^\beta + u^\alpha q^\beta) + p^\alpha u^\beta - t^{\alpha\beta} \tag{6}$$

and the electromagnetic energy-momentum tensor, following GROT/ERINGEN [3], by:

$$\underset{\text{em}}{T^{\alpha\beta}} = -F^{\alpha\nu} G^\beta_\nu + \frac{1}{4\mu_0} F^{\mu\nu} F_{\nu\mu} g^{\alpha\beta} . \tag{7}$$

The dissipation inequality reads

$$n_0 D\eta + \Phi^\alpha_{,\alpha} \geqq h , \tag{8}$$

where n_0 is the particle density in a local rest frame, $\boldsymbol{u}$ the 4-velocity, e the specific energy density, $\boldsymbol{q}$ the heat flux, $\boldsymbol{p}$ the nonmechanical momentum, $\boldsymbol{t}$ the 4-stress tensor, $\boldsymbol{g}$ the metric, $\boldsymbol{F}$ and $\boldsymbol{G}$ are the electromagnetic field tensors, $\boldsymbol{j}$ the charge-current 4-vector, $\boldsymbol{\varepsilon}$ the total antisymmetric tensor, $D = u^\alpha \partial/\partial x^\alpha$ the covariant material time-derivative, η the specific density of entropy, $\boldsymbol{\Phi}$ the conductive flux of entropy and h the supply of entropy in a local rest frame. An appropriate state space for heat conducting viscous fluids in external electromagnetic fields is given by

$$X := (n_0; u^\alpha; u^\alpha_{,\beta}; \theta; \theta_{,\beta}; \mathscr{E}^\alpha; \mathscr{B}^\alpha) , \tag{9}$$

where θ is the scalar temperature in a local rest frame, $(\mathscr{E}^\alpha)$ and $(\mathscr{B}^\alpha)$ are the electric 4-field and the magnetic 4-flux, respectively [3]. Using Liu's lemma for exploiting the dissipation inequality we get the general form of the material equations. For brevity only one aspect of our results is discussed here.

It is well known that there are liquids wich show birefringence in shear flow. Therefore, their polarisation must depend on the gradient of velocity. The exploitations of the dissipation inequality for liquids in electromagnetic fields discussed in the literature either neglect the gradient of velocity [5], or use the method of COLEMAN and NOLL [1] and get the results [3]:

$$\left[n_0 \frac{\partial \psi}{\partial \mathscr{E}_\alpha} = -\mathscr{P}^\alpha \wedge \frac{\partial \psi}{\partial\, \partial_\gamma u^\nu} = 0 \right] \Rightarrow \frac{\partial \mathscr{P}^\alpha}{\partial\, \partial_\gamma u_\nu} = 0 \tag{10}$$

($\psi := e - \theta\eta$ is the specific free energy density, $(\mathscr{P}^\alpha)$ is the polarisation 4-vector). It is obvious that these results cannot describe materials with shear-induced birefringence. Using LIU's lemma [4] we prove the following statement:

$$\left[\Phi^\alpha = \theta^{-1} q^\alpha \wedge n_0 \frac{\partial \psi}{\partial \varepsilon_\alpha} = -\mathscr{P}^\alpha \right] \Rightarrow \frac{\partial \mathscr{P}^\alpha}{\partial\, \partial_\gamma u_\nu} = 0 . \tag{11}$$

Disregarding the assumption

$$\boldsymbol{\Phi} = \theta^{-1} \boldsymbol{q} , \tag{12}$$

we get material equations which allow

$$\frac{\partial \mathscr{P}^\alpha}{\partial\, \partial_\gamma u_\nu} \neq 0\,. \tag{13}$$

Therefore, it becomes possible to describe shear-induced birefringence. Deducing resonable replacements for (12) is no simple task. We found some expressions for $\boldsymbol{\Phi}$ leading to (13). Now they have to be checked for plausibility.

We conclude that the widely accepted assumption $\boldsymbol{\Phi} = \theta^{-1}\boldsymbol{q}$ is wrong, at least for liquids with shear-induced birefringence. The construction of some appropriate replacements for this relation will be finished in the near future. See also [6].

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Reciprocal Theorem for Viscoelastic Thermodiffusion in the Electromagnetic Field

1. Introduction

We will analyse the initial-boundary value problem of viscoeleasticity thermodiffusion occuring together with an external electromagnetic field.

The medium is treated as a multicomponent one. We distinguish in it a skeleton with density ϱ^0, one order higher than the diffusion density of components having density $\varrho^1, \ldots, \varrho^n$. The skeleton is dielectric, whereas the motion of the remaining components should be treated as a flow of ions relation to the skeleton.

The physically discussed problem is described by the set of equations of balance, physical equations and initial-boundary conditions.

2. Equations of process

After linearization, the set of equations of balance will assume the form

balance of mass

$$\varrho \dot{c}^\alpha = \varrho R^\alpha - \operatorname{div} \boldsymbol{j}^\alpha\,, \qquad \alpha = 0, 1, 2 \ldots n \tag{1}$$

balance of electric charge

$$\dot{\overline{\varrho e}} = -\operatorname{div}\left(\varrho e \boldsymbol{w} + \boldsymbol{J}\right) \tag{2}$$

where

$$c^\alpha = \varrho^\alpha/\varrho\,, \qquad \boldsymbol{v}^\alpha = \boldsymbol{w} + \boldsymbol{u}^\alpha\,, \qquad \varrho\boldsymbol{w} = \sum_\alpha \varrho^\alpha \boldsymbol{v}^\alpha\,, \qquad \boldsymbol{j}^\alpha = \varrho^\alpha \boldsymbol{u}^\alpha\,, \qquad \sum_\alpha \varrho R^\alpha = 0 \qquad \sum_\alpha \varrho^\alpha e^\alpha = \varrho e\,,$$

$$\sum_\alpha \varrho^\alpha e^\alpha \boldsymbol{v}^\alpha = \varrho e \boldsymbol{w} + \boldsymbol{J}\,, \qquad \boldsymbol{J} = \sum_\alpha \boldsymbol{j}^\alpha e^\alpha \tag{3}$$

balance of momentum

$$\varrho\dot{\boldsymbol{w}} = \varrho\boldsymbol{F} + \operatorname{div}\boldsymbol{\sigma} + \varrho e\left[\boldsymbol{E} + \frac{1}{c}\boldsymbol{w}\times\boldsymbol{B}\right] + \frac{1}{c}\boldsymbol{J}\times\boldsymbol{B}\,, \qquad \varrho\boldsymbol{F} = \sum_\alpha \varrho^\alpha \boldsymbol{F}^\alpha\,, \qquad \boldsymbol{\sigma} = \sum_\alpha \sigma^\alpha\,, \qquad \sum_\alpha \varrho^\alpha \boldsymbol{u}^\alpha \times \boldsymbol{u}^\alpha \approx 0 \tag{4}$$

reduced energy balance

$$\varrho T_0 \dot{S} = \varrho r - \operatorname{div}\boldsymbol{q} + \varrho e \boldsymbol{\omega}\cdot\boldsymbol{E} + c\operatorname{div}(\boldsymbol{E}\times\boldsymbol{B}) \tag{5}$$

Maxwell's equations

$$\dot{\boldsymbol{D}} + 4\pi(\varrho e\boldsymbol{w} + \boldsymbol{J}) = c\operatorname{rot}\boldsymbol{H}\,, \qquad \operatorname{div}\boldsymbol{D} = 4\pi\varrho e \tag{6}$$

$$\dot{\boldsymbol{B}} = -c\operatorname{rot}\boldsymbol{E}\,, \qquad \operatorname{div}\boldsymbol{B} = 0\,. \tag{7}$$

In the above equations the symbols ϱ, ϱ^α, ϱR^α, $\boldsymbol{j}^\alpha$, $\boldsymbol{v}^\alpha$, $\boldsymbol{w}$, $\boldsymbol{u}^\alpha$, $\varrho^\alpha e^\alpha$, $\varrho e\boldsymbol{w}$, $\boldsymbol{J}$, $\varrho\boldsymbol{F}$, $\boldsymbol{\sigma}^\alpha$, ε, $\boldsymbol{d}$, $\boldsymbol{E} = -\operatorname{grad}\varphi$, $\boldsymbol{H}$, $\boldsymbol{D}$, B, c, T, ϱS, ϱr, $\boldsymbol{q}$, M^α denotes respectively the mass density of the component α and all the medium, the source and the flux of mass, the velocity of the component, barycenter and diffusion velocity, the charge of the component α, convection and diffusion component of electric current, mass forces, tensor of stress and strain, electric and magnetic field, electric and magnetic induction, light velocity, temperature, entropy, source and flux of heat and chemical potential.

The second relations group comprises constitutive equations for

stress tensor $\boldsymbol{\sigma}$

$$\boldsymbol{\sigma} = \boldsymbol{G} * \mathrm{d}\varepsilon - \boldsymbol{\Phi} * \mathrm{d}\theta - \sum_\alpha \boldsymbol{\varphi}^\alpha * \mathrm{d}c^\alpha - \boldsymbol{A} * \mathrm{d}\boldsymbol{E} - \boldsymbol{Q} * \mathrm{d}\boldsymbol{B} \tag{8}$$

entropy ϱS

$$-\varrho S = \boldsymbol{\Phi} * \mathrm{d}\varepsilon + m * \mathrm{d}\theta + \sum_\alpha l^\alpha * \mathrm{d}c^\alpha + \tilde{\boldsymbol{F}} * \mathrm{d}\boldsymbol{E} + \boldsymbol{N} * \mathrm{d}\boldsymbol{B} \tag{9}$$

chemical potential M^α of the component α

$$M^\alpha = -\boldsymbol{\varphi}^\alpha * \mathrm{d}\varepsilon + l^\alpha * \mathrm{d}\theta + n^{\alpha} * \mathrm{d}c^\alpha + \boldsymbol{C}^\alpha * \mathrm{d}\boldsymbol{E} + \boldsymbol{R}^\alpha * \mathrm{d}\boldsymbol{B} \tag{10}$$

electric induction $\boldsymbol{D}$

$$\boldsymbol{D} = -\boldsymbol{A} * \mathrm{d}\varepsilon + \tilde{\boldsymbol{F}} * \mathrm{d}\theta + \sum_\alpha \boldsymbol{C}^\alpha * \mathrm{d}c^\alpha + \boldsymbol{K} * \mathrm{d}\boldsymbol{E} + \boldsymbol{L} * \mathrm{d}\boldsymbol{B} \tag{11}$$

magnetic field $\boldsymbol{H}$

$$\boldsymbol{H} = -\boldsymbol{Q} * \mathrm{d}\varepsilon + \boldsymbol{N} * \mathrm{d}\theta + \sum_\alpha \boldsymbol{R}^\alpha * \mathrm{d}c^\alpha + \boldsymbol{L} * \mathrm{d}\boldsymbol{E} + \boldsymbol{\Pi} * \mathrm{d}\boldsymbol{B} \tag{12}$$

heat flux q

$$\boldsymbol{q} = -k\operatorname{grad}\theta\,, \qquad \theta = t - T_0 \tag{13}$$

mass flux $\boldsymbol{j}^\alpha$ and diffusion current $\boldsymbol{J}$

$$\boldsymbol{j}^\alpha = -D^\alpha \operatorname{grad} M^\alpha\,, \qquad \boldsymbol{J} = \sum_\alpha e^\alpha \boldsymbol{j}^\alpha\,. \tag{14}$$

In these equations $2\dot{\varepsilon} = \operatorname{grad}\boldsymbol{w} + (\operatorname{grad}\boldsymbol{w})^{\mathrm{T}}$ denotes the linear tensor of strain velocity whereas the symbols G_{ijkl}, Φ_{ij}, φ^α_{ij}, A_{ijk}, Q_{ijk}, m, l^α, $\tilde{F}_i$, N_i, l^α, n^α, C^α_i, R^α_i, L_{ij}, K_{ij}, Π_{ij} denote the relaxations functions in thermodiffusion process and $*$ Stieltjes' convolution.

It has been assumed that the described medium should be treated as a multicomponent viscoelasticity body.

The third group comprises the initial-boundary conditions and assumes the form:

$$\begin{aligned} &\boldsymbol{\sigma}\cdot\boldsymbol{n}|_{A_1} = \boldsymbol{P}\,, \qquad \boldsymbol{w}|_{A_2} = \bar{\boldsymbol{w}}\,, \qquad \boldsymbol{j}^\alpha|_{A_3} = \bar{\boldsymbol{j}}^\alpha\,, \qquad \boldsymbol{q}|_{A_4} = \bar{\boldsymbol{q}}\,, \\ &\boldsymbol{J}|_{A_3} = \bar{\boldsymbol{J}}\,, \qquad \boldsymbol{n}\times(\boldsymbol{E} - \bar{\boldsymbol{E}})|_{A_5} = \boldsymbol{0}\,, \qquad \boldsymbol{H}|_{A_5} = \bar{\boldsymbol{H}} \text{ and } \boldsymbol{n}\times(\boldsymbol{H} - \bar{\boldsymbol{H}})|_{A_6} = \boldsymbol{0}\,, \qquad \boldsymbol{E}|_{A_6} = \bar{\boldsymbol{E}} \end{aligned} \tag{15}$$

$$\boldsymbol{w}(0_+) = \mathring{\boldsymbol{w}}\,, \qquad c^\alpha(0_+) = \mathring{c}^\alpha\,, \qquad \theta(0_+) = \mathring{\theta}\,, \qquad \boldsymbol{E}(0_+) = \mathring{\boldsymbol{E}}\,, \qquad \boldsymbol{H}(0_+) = \mathring{\boldsymbol{H}}\,. \tag{16}$$

3. Reciprocal theorem

We will analyse two sets of fields

$$[\boldsymbol{\sigma}, \boldsymbol{\varepsilon}, \boldsymbol{P}, \varrho^S, \theta, \boldsymbol{q}, M^\alpha, c^\alpha, \boldsymbol{j}^\alpha, \boldsymbol{E}, \boldsymbol{D}, \boldsymbol{H}, \boldsymbol{B}]^{1,2} \tag{17}$$

affecting the same medium.

The quantities determining these fields fulfil the physical equations, the balance equations and initial-boundary conditions mentioned above. From the properties of symmetry of the afore said equations we receive the sought reciprocal theorem.

$$\begin{aligned}
&\int_{A_1} (\boldsymbol{P}_1 * \mathrm{d}\boldsymbol{w}_2 - \boldsymbol{P}_2 * \mathrm{d}\boldsymbol{w}_1)\,\mathrm{d}A + \int_V (\varrho \boldsymbol{F}_1 * \mathrm{d}\boldsymbol{w}_2 - \varrho \boldsymbol{F}_2 * \mathrm{d}\boldsymbol{w}_1)\,\mathrm{d}V - \\
&- \int_{A_6} (\varrho e_1 \varphi_1 * \mathrm{d}\boldsymbol{w}_2 - \varrho e_2 \varphi_2 * \mathrm{d}\boldsymbol{w}_1)\,\boldsymbol{n}\,\mathrm{d}A + \int_V (\varrho e_1 \varphi_1 * \boldsymbol{\varepsilon}_2 - \varrho e_2 \varphi_2 * \boldsymbol{\varepsilon}_1)\,\mathrm{d}V + \\
&+ \int_V (\mathring{\boldsymbol{w}}_1\,\mathrm{d}\boldsymbol{w}_2 - \mathring{\boldsymbol{w}}_2\,\mathrm{d}\boldsymbol{w}_1)\,\mathrm{d}V + \\
&+ \int_{A_4} \frac{1}{\varrho T_0 k} (\bar{\boldsymbol{q}}_1 * \theta_2 - \bar{\boldsymbol{q}}_2 * \theta_1)\,\boldsymbol{n}\,\mathrm{d}A + \int_V \frac{1}{T_0} (r_1 * \theta_2 - r_2 * \theta_1)\,\mathrm{d}V + \\
&+ \int_{A_3} \sum_\alpha \frac{1}{\varrho D^\alpha} (\bar{\boldsymbol{j}}_1^\alpha * M_2^\alpha - \bar{\boldsymbol{j}}_2^\alpha * M_1^\alpha)\,\boldsymbol{n}\,\mathrm{d}A + \int_V \sum_\alpha (R_1^\alpha * M_2^\alpha - R_2^\alpha * M_1^\alpha)\,\mathrm{d}V + \\
&+ \int_V (\mathring{S}_1 \theta_2 - \mathring{S}_2 \theta_1)\,\mathrm{d}V + \int_V \sum_\alpha (\mathring{c}_1^\alpha M_2^\alpha - \mathring{c}_2^\alpha M_1^\alpha)\,\mathrm{d}V - \\
&- \int_{A_3} [(\varrho e_1 \boldsymbol{w}_1 + \bar{\boldsymbol{J}}_1) * \varphi_2 - (\varrho e_2 \boldsymbol{w}_2 + \bar{\boldsymbol{J}}_2) * \varphi_1]\,\boldsymbol{n}\,\mathrm{d}A + \\
&+ \int_V (\dot{\overline{\varrho e}}_1 * \varphi_2 - \dot{\overline{\varrho e}}_2 * \varphi^1)\,\mathrm{d}V + \int_V [(\mathring{\boldsymbol{E}}_2 \boldsymbol{E}_1 - \mathring{\boldsymbol{E}}_1 \boldsymbol{E}_2) + (\mathring{\boldsymbol{H}}_1 \boldsymbol{H}_2 - \mathring{\boldsymbol{H}}_2 \boldsymbol{H}_1)]\,\mathrm{d}V + \\
&+ \int_{A_6} \{\bar{\boldsymbol{E}}_2 * [\boldsymbol{n} \times (\boldsymbol{H}_1 - \bar{\boldsymbol{H}}_1)] - \bar{\boldsymbol{E}}_1 * [\boldsymbol{n} \times (\boldsymbol{H}_2 - \bar{\boldsymbol{H}}_2)]\}\,\mathrm{d}A + \\
&+ \int_{A_5} \{\bar{\boldsymbol{H}}_1 * [\boldsymbol{n} \times (\boldsymbol{E}_2 - \bar{\boldsymbol{E}}_2)] - \bar{\boldsymbol{H}}_2 * [\boldsymbol{n} \times (\boldsymbol{E}_1 - \bar{\boldsymbol{E}}_1)]\}\,\mathrm{d}A = 0\,.
\end{aligned} \tag{18}$$

The obtained dependence includes as a particular case the reciprocal theorem for non-coupled, mechanical, heat, diffusion and electromagnetic problems.

This theorem also permits the construction approximate methods of solving initial-boundary problems of thermodiffusion in the electromagnetic field.

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