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The Correspondence Between Equations of Thermodiffusion and Theory of Mixtures*

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With 2 Figures

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Summary

The balance equations of mass, momentum, energy and entropy have been studied in the paper. A comparison has been made between these equations and equations describing conjugate thermodiffusion flows in solids.

1. Introduction

The paper deals with description on that class of mass flows with reference to skeleton which can be approximated by equations of thermodiffusion in solid. We will use the equations of mass, momentum, angular momentum, energy and entropy inequality balances, given in typical form for the theory of mixture [1]. All above equations will be analysed for using them in description of flows closed with classical diffusion in solid. So we will present the balances for each component and for mixture. In the balances for each component, in the source term we must account for the influence of remaining components. However, balances for entire mixture are identical as for one-component body.

Comparing the balances for the both approaches to the problem we should underline equal rank of treating all components in the theory of mixtures. However, in thermodiffusion we analyse only one field of velocity of component marked out i.e. skeleton. As the result, in thermodiffusion we obtain the more formally simple set of equations describing the boundary problem. Certainly, the both descriptions exist undependently one to another, but we can compare them. It is also the aim of considerations.

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2. The Balances for Multi-Component Body

We will analyse the problem of exchange of mass, momentum, energy in multi-component body and we will demand the satisfaction of the balances, as in the theory of mixtures. We will present the balances in total and local form respectively, for every component and for entire mixture, starting from the mass balances and ending on the inequalities of entropy. We use the traditional notations.

The mass balances have the following form:

— the balance for component α $(\alpha = 0, 1, \ldots, n)$

$$\frac{d}{dt}\int\limits_V \varrho^\alpha \, dV = \int\limits_V \varrho R^\alpha \, dV \to \frac{\partial \varrho^\alpha}{\partial t} + (\varrho^\alpha v_k^\alpha)_{,k} = \varrho R^\alpha, \qquad (\,)_{,k} \equiv \frac{\partial}{\partial x_k} \tag{2.1}$$

— the balance for entire mixture

$$\begin{aligned} \frac{d}{dt}\int\limits_V \varrho \, dV = 0 \to \frac{\partial}{\partial t}(\varrho^0 + \varrho^1 + \cdots + \varrho^n) &+ (\varrho^0 v_k^0 + \varrho^1 v_k^1 + \cdots + \varrho^n v_k^n) \\ &= \varrho(R^0 + R^1 + \cdots + R^n) = 0, \end{aligned} \tag{2.2}$$

$$\frac{\partial \varrho}{\partial t} + (\varrho w_k)_{,k} \equiv 0,$$

$$\sum_\alpha R^\alpha = 0, \qquad \varrho w_k = \sum_\alpha \varrho^\alpha v_k^\alpha, \qquad v_k^\alpha = w_k + u_k^\alpha, \qquad \sum_\alpha \varrho^\alpha u_k^\alpha = 0.$$

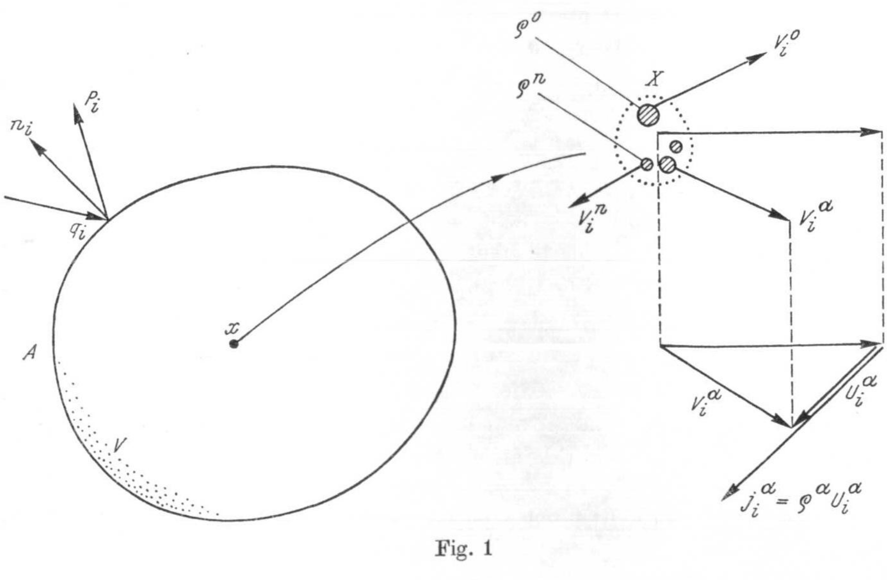

Fig. 1

After decomposing the velocity v_k of component α on sum of the average velocity w_k and the increment u_k^α we obtain

$$\frac{\partial \varrho^\alpha}{\partial t} + [\varrho^\alpha(w_k + u_k^\alpha)]_{,k} = \varrho R^\alpha . \tag{2.3}$$

The equation of equal rank is obtained after introducing the concentration $c^\alpha = \dfrac{\varrho^\alpha}{\varrho}$ into (2.3)

$$\begin{aligned} &\varrho \frac{dc^\alpha}{dt} + (\varrho^\alpha u_i^\alpha)_{,i} = \varrho R^\alpha , \\ &\varrho \frac{d\bar{c}_{ii}^\alpha}{dt} + (j_i^\alpha)_{,i} = \varrho \bar{R}_{ii}^\alpha , \qquad j_i^\alpha = \varrho^\alpha u_i^\alpha . \end{aligned} \tag{2.4}$$

It will be used to formulating the correspondence between the balances of thermodiffusion and the theory of mixtures. The momentum balance for the component (α) has the form

$$\frac{d}{dt} \int_V \varrho^\alpha v_i^\alpha \, dV = \int_V (\varrho^\alpha F_i^\alpha + \Phi_i^\alpha) \, dV + \int_A P_i^\alpha \, dA \tag{2.5}$$

and for entire mixture

$$\begin{aligned} &\sum_\alpha \frac{d}{dt} \int_V \varrho^\alpha v_i^\alpha \, dV = \sum_\alpha \int_V (\varrho^\alpha F_i^\alpha + \Phi_i^\alpha) \, dV + \sum_\alpha \int_A t_{ij}^\alpha n_j \, dA \\ &\varrho \frac{dw_i}{dt} = \varrho F_i + \left(t_{ij} - \sum_\alpha \varrho^\alpha u_i^\alpha u_j^\alpha \right)_{,j} \end{aligned} \tag{2.6}$$

where

$$\varrho F_i = \sum_\alpha \varrho^\alpha F_i^\alpha , \qquad \sum_\alpha \Phi_i^\alpha = 0, \qquad t_{ij} = \sum_\alpha t_{ij}^\alpha , \qquad t_{ij} = t_{ji} .$$

The energy balance has the following form for entire mixture

$$\begin{aligned} \sum_\alpha \frac{d}{dt} \int_V \varrho^\alpha (U^\alpha + K^\alpha) \, dV = &\sum_\alpha \int_V (\varrho^\alpha r^\alpha + \varrho^\alpha F_i v_i^\alpha + E^\alpha) \, dV \\ &+ \sum_\alpha \int_A (t_{ij}^\alpha v_j^\alpha + q_i^\alpha) \, n_i \, dA . \end{aligned} \tag{2.7}$$

After using the previous balances and introducing the following quantifies

$$\varrho U = \sum_\alpha \varrho^\alpha U^\alpha , \qquad \varrho K = \sum_\alpha \varrho^\alpha K^\alpha , \qquad \varrho r = \sum_\alpha \varrho^\alpha r^\alpha ,$$

$$q_i = \sum_\alpha q_i^\alpha , \qquad \sum_\alpha E^\alpha = 0$$

we obtain

$$\varrho \frac{d}{dt}(U+K) = \varrho r + \varrho F_i w_i + \sum_\alpha \varrho^\alpha F_i^\alpha u_i^\alpha - q_{i,i} + (t_{ij}w_j)_{,i} + \sum_\alpha [(t_{ij}^\alpha u_j^\alpha) - \varrho^\alpha u_i^\alpha U^\alpha - \varrho^\alpha u_i^\alpha K^\alpha]_{,i}. \tag{2.8}$$

In the above expression

$$\sum_\alpha \varrho^\alpha F_i^\alpha v_i^\alpha = \sum_\alpha \varrho^\alpha F_i^\alpha w_i + \sum_\alpha \varrho^\alpha F_i^\alpha u_i^\alpha = \varrho F_i w_i + \sum_\alpha \varrho^\alpha F_i^\alpha u_i^\alpha$$
$$\sum_\alpha t_{ij}^\alpha v_j^\alpha = t_{ij} w_j + \sum_\alpha t_{ij}^\alpha u_j^\alpha. \tag{2.9}$$

The entropy balance is as following for entire mixture

$$\sum_\alpha \frac{d}{dt}\int_V \varrho^\alpha S^\alpha \, dV \geqq \sum_\alpha \int_V \varrho^\alpha \frac{r^\alpha}{T}\, dV - \sum_\alpha \int_A \left(\frac{q_i}{T}\right) n_i \, dA \rightarrow$$
$$\varrho \frac{dS}{dt} \geqq \frac{\varrho r}{T} - \left(\frac{q_i}{T}\right)_{,i} - \sum_\alpha (\varrho^\alpha u_i^\alpha S^\alpha)_{,i} \tag{2.10}$$
$$\varrho S = \sum_\alpha \varrho^\alpha S^\alpha, \qquad \varrho r = \sum_\alpha \varrho^\alpha r^\alpha, \qquad q_i = \sum_\alpha q_i^\alpha.$$

The typical marks are used in the equations listed above. In particularly: ϱ, ϱ^α, w_i, v_i^α, u_i^α, R^α, $\varrho^\alpha F_i^\alpha$, Φ_i^α, t_{ij}^α, $\varrho^\alpha U^\alpha$, $\varrho^\alpha K^\alpha$, $\varrho^\alpha S^\alpha$, $\varrho^\alpha r^\alpha$, q_i^α, denote respectively: density of entire mixture, density of component (α), average velocity, velocity of component α, increment of velocity with respect to average value, mass source, mass force, transfer of momentum from remaining components, stress tensor of component, internal energy, kinematic energy, entropy, heat source, heat flux of component (α).

3. The Comparison of Theories

After presumption the equality of mass forces, internal and kinetic energies, entropies of mass unit for each component

$$F_0 = F_1 = \cdots = F_n, \qquad K_1 \cong K_2 \cong \ldots \cong K_n, \qquad S_1 \cong S_2 \cong \ldots \cong S_n \tag{3.1}$$

the formal form of the balances in theory of mixtures leads to equations related to thermodiffusion in solid state. They have the following form:

$$\varrho \frac{dc^\alpha}{dt} = \varrho R^\alpha - (j_i^\alpha)_{,i}, \qquad j_i^\alpha = \varrho^\alpha u_i^\alpha$$
$$\varrho \frac{dw_i}{dt} = \varrho F_i + \left[t_{ij} - \sum_\alpha \varrho^\alpha u_i^\alpha u_j^\alpha\right]_{,j}$$
$$\varrho \frac{d}{dt}(U+K) = \varrho r + \varrho F_i w_i + [t_{ij} w_j - q_i]_{,i} - \sum_\alpha [j_i^\alpha U^\alpha - t_{ij}^\alpha u_j^\alpha]_{,i} \tag{3.2}$$
$$\varrho \frac{dS}{dt} \geqq \frac{\varrho r}{T} - \left(\frac{q_i}{T}\right)_{,i}.$$

Let analyse the case $\sum_\alpha \varrho^\alpha u_i^\alpha u_j^\alpha \cong 0$ two possibilities arise:

($\mathscr{E}_1$) one marked out component (i.e. skeleton) ϱ^0 occurs, and it arises for this case

$$[\varrho^0 \gg \varrho^\beta;\ \beta = 1, 2, \ldots, n] \Rightarrow \varrho w_i \cong \varrho^0 v_i^0$$

$$[\varrho^\alpha v_i^\alpha - \varrho^\beta v_i^\beta = 0;\ \alpha = 1, 3, 5, \ldots, \beta = 2, 4, 6, \ldots] \Rightarrow \varrho w_i = \varrho^0 v_i^0$$

it denotes that the average velocity ϱw_i is approached to velocity of component of density ϱ^0. It occurs during migration of dissipated component in skeleton.

($\mathscr{E}_2$) a few components of compareable masses $\varrho^\beta \sim \varrho^0 \sim \varrho^1 \cdots \sim \varrho^k$ and identical velocities $v_i^0 = \cdots = v_i^k$ exist and it arise

$$[\varrho^\beta \gg \varrho^\delta;\ \beta = 0, 1, 2, \ldots, k,\ \delta = k+1, k+2, \ldots, n] \Rightarrow \varrho w_i \cong \left(\sum_\beta \varrho^\beta\right) v_i^0.$$

Now if we achieve the substitution as following

$$\begin{aligned}
&\sum_\alpha (t_{ij}^\alpha u_i^\alpha)_{,j} = \sum_\alpha \left(\frac{t_{ij}^\alpha}{\varrho^\alpha}\varrho^\alpha u_i^\alpha\right)_{,j} \to -\sum_\alpha (N_{ij}^\alpha j_i^\alpha)_{,j}\\
&\sum_\alpha (-\varrho^\alpha u_i^\alpha U^\alpha)_{,i} = \sum_\alpha (-\varrho^\alpha u_i^\alpha \delta_{ij} U^\alpha)_{,j} \to -\sum_\alpha (L_{ij}^\alpha j_i^\alpha)_{,j}\\
&-\sum_\alpha (\varrho^\alpha u_i^\alpha \delta_{ij} U^\alpha - t_{ij}^\alpha u_i^\alpha)_{,j} = -\sum_\alpha \left[j_i^\alpha\left(\delta_{ij}U^\alpha - \frac{t_{ij}^\alpha}{\varrho^\alpha}\right)\right]_{,j}\\
&= \sum_\alpha [-j_i^\alpha(L_{ij}^\alpha + N_{ij}^\alpha)]_{,j} = -\sum_\alpha (j_i^\alpha M_{ij}^\alpha)_{,j}\\
&\frac{t_{ij}^\alpha}{\varrho^\alpha} = N_{ij}^\alpha, \qquad \delta_{ij}U^\alpha = L_{ij}^\alpha, \qquad M_{ij}^\alpha = \delta_{ij}U^\alpha + N_{ij}^\alpha
\end{aligned} \tag{3.3}$$

we will gain the full correspondence between the equations of theory of mixtures and thermodiffusion. Also we will obtain the limitations which are laid on descriptions of component migration in thermodiffusion in solid body. Ultimately we have the set of equations

$$\begin{aligned}
&\varrho \frac{dc^\alpha}{dt} = \varrho R^\alpha - (j_i^\alpha)_{,i}, \qquad j_i^\alpha = \varrho^\alpha u_i^\alpha,\\
&\varrho \frac{dw_i}{dt} = \varrho F_i + (t_{ij})_{,j}, \qquad t_{ij} = t_{ji},\\
&\varrho \frac{d}{dt}(U + K) = \varrho r + \varrho F_i w_i + \left(t_{ij} w_j - q_i - \sum_\alpha j_j^\alpha M_{ji}^\alpha\right)_{,i}\\
&\varrho \frac{dS}{dt} \geq \frac{\varrho r}{T} - \left(\frac{q_i}{T}\right)_{,i}.
\end{aligned} \tag{3.4}$$

The above set of equations let us describe thermodiffusional processes in solid body after completion by physical equations and boundary conditions.

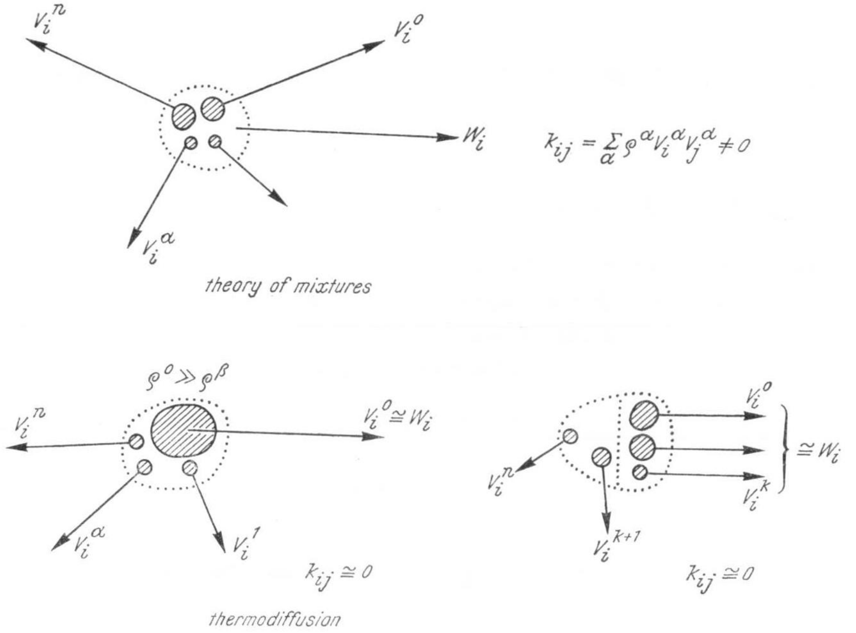

Fig. 2 .

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[1] Green, A. E., Laws, N.: Global properties of mixtures. Arch. Rat. Mech. and Anal. 1, 45–61 (1971).
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